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Construction of Quartic Graphs

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It is shown that all quartic graphs can be constructed successively from a K_5 by applying two types of operations called H-type and V-type expansions.

It is also shown that the two types of operations are necessary to successively construct a regular graph of an even degree from a complete graph of the same degree.

1. INTRODUCTION

Properties of cubic graphs have been investigated by a number of people. Among them is Johnson's work [1] (see also Ore [2]) on construction and reduction of cubic graphs. According to Johnson, a cubic graph can be reduced to another cubic graph with a smaller number of vertices by an operation called H-reduction. Also a cubic graph can be expanded to another cubic graph with a larger number of vertices. Recently Toida [3] modified Johnson's results to find a method of constructing a planar cubic graph from another planar cubic graph with a smaller number of vertices. These works suggest the problems of how to reduce and/or construct regular graphs of a general degree.

In this paper as a first step toward the generalization the problems are solved for quartic graphs, i.e., regular graphs of degree 4. Some of the results on quartic graphs are generalized for regular graphs of even degrees.

2. PRELIMINARIES

We list symbols which are used often in this paper:

G_{2n}^m is a simple regular undirected graph of degree $2n$ with m vertices where $n \geq 2$.

If u and v are vertices, then uv is the edge (no parallel edges are allowed) between u and v and $\{u, v\}$ is a set of vertices.

K_m is a complete graph with m vertices.

If G is a graph and V is a set of vertices of G, then $G - V$ is the graph obtained from G by removing all the vertices of V and the edges connected to them.

3. Reduction of Quartic Graphs

Given a G_{2n}^m, there are a number of ways of reducing it to a G_{2n}^k where $k < m$. Let us consider two among them:

(1) *V-type reduction.* Let v be a vertex of G_{2n}^m. Then v has $2n$ adjacent vertices. Let $v_1, v_2, ..., v_{2n}$ be the adjacent vertices of v. Suppose that $v_1, v_2, ..., v_{2n}$ can be paired such that each of the pairs is not connected by an edge in G_{2n}^m. For example $\{v_1, v_2\}, \{v_3, v_4\}, ..., \{v_{2i-1}, v_{2i}\}, ..., \{v_{2n-1}, v_{2n}\}$ are such pairs if there are no edges $v_{2i-1}v_{2i}$ in G_{2n}^m where $i = 1, 2, ..., n$. Then we can produce G_{2n}^{m-1} from G_{2n}^m by removing the vertex v and its edges and by adding an edge between each of the pairs of the vertices. In the example $(G_{2n}^m - \{v\}) \cup (\bigcup_{i=1}^n \{v_{2i-1}v_{2i}\})$ is a desired G_{2n}^{m-1}. We call this type of reduction a *V-type reduction.*

(2) *H-type reduction.* Let u and v be a pair of vertices of G_{2n}^m connected by an edge. Then there are $2n$ vertices connected to u and $2n$ vertices connected to v.

Let $u, v_1, v_2, ..., v_{2n-1}$ be the $2n$ vertices connected to v and let $v, u_1, u_2, ..., u_{2n-1}$ be the $2n$ vertices connected to u. Some of the u_i's may be identical to some of the v_j's, where $i, j = 1, 2, ..., 2n - 1$. Since a G_{2n}^m is simple, no two of u_i's (or v_j's) can be identical. Hence any u_i cannot be identical to more than one v_j.

Suppose that $u_1, u_2, ..., u_{2n-1}, v_1, v_2, ..., v_{2n-1}$ can be paired such that

(1) each pair consists of two distinct vertices,

(2) the vertices in any pair are not connected by an edge of G_{2n}^m,

(3) each of u_i's (v_i's) appears once and only once in the pairs unless it is identical to a v_j (a u_j). If a u_k is identical to a v_l then u_k is paired with two distinct vertices to form two pairs.

Suppose such a pairing is possible, then we can produce a G_{2n}^{m-2} from G_{2n}^m by removing the vertices u and v and their edges and by adding an edge between the vertices of each pair.

For example, suppose that, in a G_{2n}^m, v, u_1, u_2, and u_3 are adjacent to u, u, v_1, v_2, and v_3 are adjacent to v, u_1 is identical to v, u_1 is identical to v_1, all other vertices are distinct, edges u_2v_2 and u_3v_3 exist, and no other edges exist among u's and v's. Then clearly (u_1, u_2), (u_1, v_3), and (u_3, v_2) is a pairing which satisfies the above-mentioned (1)–(3). Hence we can produce a G_{2n}^{m-2} from G_{2n}^m by removing vertices u and v and their edges and by adding edges u_1u_2, u_1v_3, and u_3v_2, that is,

$$(G_{2n}^m - \{u, v\}) \cup \{u_1u_2, u_1v_3, u_3v_2\}$$

is a G_{2n}^{m-2}.

We call this type of reduction an *H-type reduction*.

We say G_{2n}^m is *H-(or V)-irreducible* if it cannot be reduced to a G_{2n}^{m-2} (or G_{2n}^{m-1}) by an H- (or V)-type reduction.

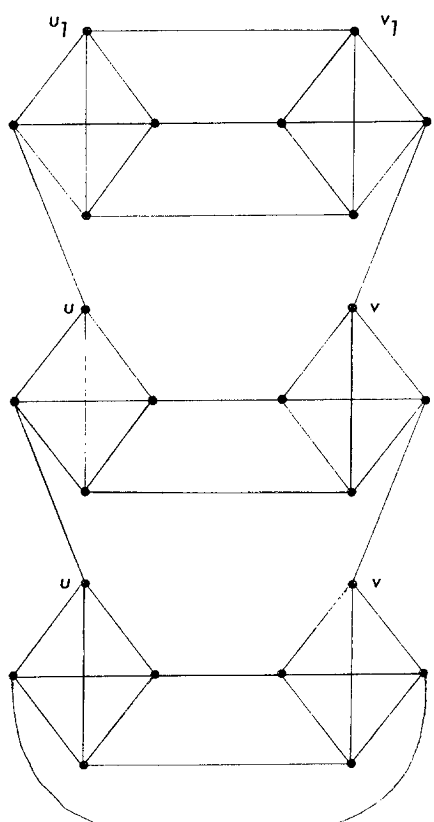


FIGURE 1

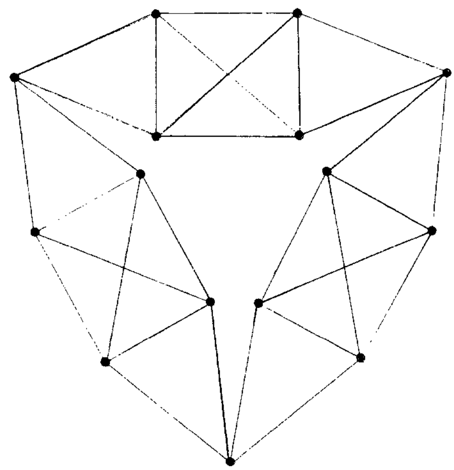


FIGURE 2

The graph of Figure 1 is V-irreducible and the graph of Figure 2 is H-irreducible. It can easily be shown that there are infinitely many V-irreducible and H-irreducible graphs among simple regular graphs of even degrees.

Thus an H-type or a V-type reduction alone cannot always be used to produce a regular graph of an even degree from a given regular graph of the same degree. A natural question then is whether one can always apply at least one of the two types of reduction to the given graph. Though we do not say anything definite on general regular graphs, we show in the following that this is the case for connected quartic graphs except K_5. That is, any connected quartic graph can be reduced to K_5 by applying H-type and/or V-type reductions.

We first investigate the cases in which a vertex of a quartic graph cannot be eliminated by a V-type reduction.

DEFINITION 1. A *star* consists of four vertices and three edges such that one of the vertices is connected to each of the other three vertices by an edge.

A delta is a complete graph with three vertices plus an isolated vertex.

In the following, a "graph" means a simple regular graph of degree 4.

DEFINITION 2. Let v be a vertex of a graph G.

If there is a subgraph in G which is a star (a delta) and the four vertices adjacent to v in G are in the subgraph we say that v *is connected to the star* (*the delta*).

LEMMA 1. *If a vertex, call it v, of a graph G cannot be removed by a V-type reduction, then v is connected to either a star or a delta.*

Proof. Suppose that v is connected to vertices w, x, y, and z and that these form neither a star nor a delta in G. Since v cannot be removed by a V-type reduction, at least one of each of the pairs of edges $\{wx, yz\}$, $\{wy, xz\}$, and $\{wz, xy\}$ is in G. Without loss of generality assume that wx and wy are edges of G. Since, w, x, y, and z do not form a delta, the edge xy is not in G. Furthermore, since w, x, y, and z do not form a star either, the edge wz is not in G. But one of the edges xy and wz must be in G. This is a contradiction. Thus we have the lemma.

The following theorem is one of the main results of this paper.

DEFINITION 3. A B-graph is a graph obtained by connecting two K_4's by an edge. See Figure 3.

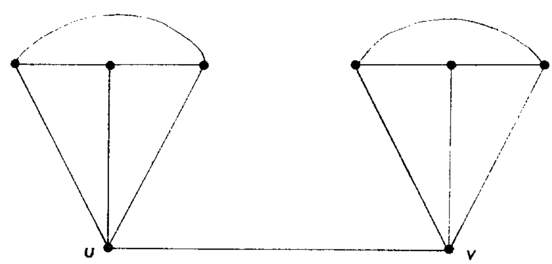


FIGURE 3

THEOREM 1. *If a connected graph G is V-irreducible, then either G is K_5 or G contains a B-graph as a subgraph.*

Proof. Let v be a vertex of G. Since v cannot be removed from G by a V-type reduction, v is connected to either a delta or a star.

Case 1. *v is connected to a delta.* Let w be the isolated vertex of the delta to which v is connected. Since G is V-irreducible, the vertex w is also connected to either a delta or a star.

(1) *w is connected to a delta.* Consider the following four cases:

(a) w is adjacent to v and to no other vertices of the delta to which v is connected.

(b) w is adjacent to v and to one of the vertices of the delta to which v is connected.

(c) w is adjacent to v and to two of the vertices of the delta to which v is connected.

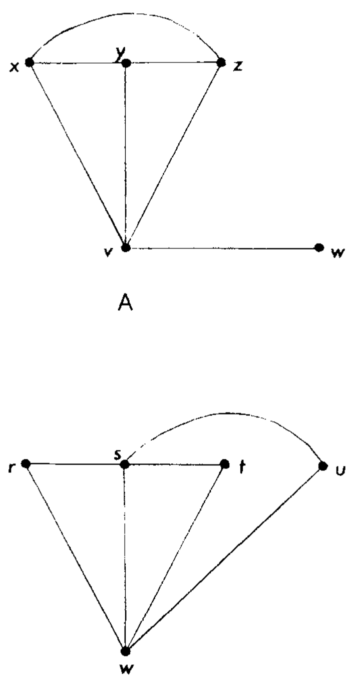
(d) w is adjacent to v and to three of the vertices of the delta to which v is connected.

In (a) there is obviously a B-graph. Case (b) cannot happen. Case (d) clearly gives us a K_5 as G.

In case (c) let p be the fourth vertex adjacent to w. Since p cannot be in a star, p must be connected to a delta. If p is adjacent to one of the vertices of the delta to which v is connected, then p can be removed from G by a V-type reduction. Hence p is not adjacent to any of the vertices of the delta except w. Thus G contains a B-graph in case (c).

Hence the theorem is true in this case.

(2) *w is connected to a star.* In this case we show that G is a K_5.



A

B

FIGURE 4

First we name the vertices of a star and a delta as in Figure 4 where v and w in Figure 4A are the v and the w of G. Since w is connected to a star, w in Figure 4A must be identical to w in Figure 4B and v must be identical to one of r, s, t, and u of Figure 4B.

If v is identical to s, then $\{x, y, z\}$ must be identical to $\{r, t, u\}$. Hence w of Figure 4A must be adjacent to each of the vertices x, y, and z. Thus we have a K_5 in this case.

If v is identical to r, t, or u, then one of the vertices x, y, and z must be identical to s. Without loss of generality let us assume that v is identical to r and that x is identical to s. Then x must be adjacent to w and $\{y, z\}$ must be identical to $\{t, u\}$. Hence each of the vertices y and z must be adjacent to w. Thus we have a K_5 in this case.

Case 2. *v is connected to a star.* In this case all we need show is that no V-irreducible graph G exists such that G has no vertex which is connected to a delta. For if there is a vertex in G which is connected to a delta then this case reduces to Case 1.

Suppose that v is connected to vertices w, x, y, and z. Then these vertices form a star. Hence one of them, say x, is connected to all the rest and v. Since G is V-irreducible, w is connected to either a delta or a star. But no vertex of G is connected to a delta. Hence w must be connected to a star. Thus w must be connected to y and z as well. But then clearly vertices v, w, and x are connected to deltas. Thus it is impossible to construct a V-irreducible graph such that it has no vertex which is connected to a delta. Q.E.D.

As we can easily see, if a graph contains a B-graph as a subgraph it is H-reducible. (Apply H-reduction to vertices u and v, where u and v are as shown in Figure 3.) Hence we have the following theorem as a corollary to Theorem 1:

THEOREM 2. *A connected graph is either H-reducible or V-reducible if it is not a K_5.*

Thus we can say that H-type and V-type reductions are sufficient to reduce a simple quartic graph to a K_5.

4. CONSTRUCTION OF QUARTIC GRAPHS

In this section a "graph" still means a simple quartic graph.

There are obvious inverse operations of H-type and V-type reductions. We call them *H-type* and *V-type expansions*, respectively. In this section we show that H-type and V-type expansions are necessary and sufficient to construct a graph from a K_5. Since we can obtain a graph from another graph by an H-type or a V-type reduction we need only show that we can always obtain a connected graph by an H-type or a V-type reduction.

REMARK 1. In any graph the number of vertices of odd degrees is even.

REMARK 2. No quartic graph can have a bridge.

For general quartic graphs we have the following theorem.

THEOREM 3. *Every connected graph G_4^m with m vertices can be constructed either from a connected G_4^{m-1} be a V-type expansion or from a connected G_4^{m-2} by an H-type expansion.*

Proof. Since from Theorem 2 we know that G_4^m can be reduced to either G_4^{m-1} or G_4^{m-2}, all we have to do is show that this can be done in such a way that they are actually connected.

Case 1. A graph G_4^{m-1} obtained from G_4^m by a V-type reduction is not connected.

Let v be the vertex of G_4^m which is removed to obtain the G_4^{m-1}.

Let w, x, y and z be the adjacent vertices of v.

If G_4^{m-1} is not connected then $G_4^m - \{v\}$ is not connected.

Let us consider possible positions of w, x, y, and z in $G_4^m - \{v\}$.

Suppose that one of them is in one connected component of $G_4^m - \{v\}$ and that others are in different component(s) from the first. Then G_4^m must have a bridge. But by Remark 2 it cannot happen. Hence the only possibility is that two of them, say w and x, are in one connected component of $G_4^m - \{v\}$ and the remaining two are in the other. In this case we add edges wy and xz to $G_4^m - \{v\}$. Then the resultant graph is connected and it is obtained from G_4^m by a V-type reduction. Hence the theorem is true for Case 1.

Case 2. A graph G_4^{m-2} obtained from G_4^m by an H-type reduction is not connected.

Let u and v be the vertices of G_4^m which are removed to obtain G_4^{m-2}.

Let q, r, and s be the adjacent vertices of u other than v.

Let x, y, and z be the adjacent vertices of v other than u.

If G_4^{m-2} is not connected then $G_4^m - \{u, v\}$ is not connected.

As we can see, some of $\{q, r, s\}$ may be identical to some of $\{x, y, z\}$. There are three possibilities: The first one is when all of q, r, s, x, y, and z are distinct; the second one is when one of q, r, and s is identical to one of x, y, and z; and the third one is when more than one of q, r, and s are identical to the same number of x, y, and z:

(A) q, r, s, x, y, and z are all distinct. Let us consider possible positions of q, r, s, x, y, and z in $G_4^m - \{u, v\}$. Then by the same reason as for Case 1 each connected component of $G_4^m - \{u, v\}$ must contain at least two of the vertices q, r, s, x, y, and z. Suppose that a connected component of $G_4^m - \{u, v\}$ contains an odd number of them. Then Remark 1 is violated since the rest of the vertices of the component are of degree 4. Thus we have the following two possibilities:

(1) Two of them are in one connected component of $G_4{}^m - \{u, v\}$ and the rest are in the other.

(2) Each connected component of $G_4{}^m - \{u, v\}$ contains exactly two of them.

For both of the cases (1) and (2) it is easy to find a pairing of the vertices q, r, s, x, y, and z so that the resultant G_4^{m-2} is connected. Hence the theorem is true for Case 2(A).

(B) One of the vertices q, r, and s is identical to one of the vertices x, y and z.

Let us assume that r is identical to x.

By similar considerations to those for Case 2(A) we can see that there are three possibilities concerning the positions of the vertices q, r, s, y, and z:

(1) r is in one connected component of $G_4{}^m - \{u, v\}$, two of the remaining four are in a second, and the rest are in the third.

(2) r and two of q, s, y, and z are in one connected component of $G_4{}^m - \{u, v\}$ and the rest are in the other.

(3) r is in one connected component of $G_4{}^m - \{u, v\}$ and the rest are in the other.

For the cases (1) and (2) it is not difficult to find a pairing of the vertices q, r, s, y, and z so that the resultant graph is a connected G_4^{m-2}. Note that r must be paired with two distinct vertices.

For the case (3), no matter what edges we add to $G_4{}^m - \{u, v\}$ to produce a G_4^{m-2}, the G_4^{m-2} is always connected.

(C) More than one of the vertices q, r, and s are identical to the same number of vertices of x, y, and z. In this case all G_4^{m-2}'s obtained from $G_4{}^m$ by an H-type reduction are connected. Hence the theorem is true for this case.

Since all possibilities are exhausted we have the theorem.

5. Conclusion

It has been shown that a simple quartic graph can be reduced to a K_5 by H-type and/or V-type reductions and that one can be constructed by H-type and/or V-type expansions from a K_5.

For regular graphs of an arbitrary even degree it has been shown that one of the two reductions alone is not sufficient to reduce them to a complete graph of the same degree. It remains to be seen if the two are sufficient for regular graphs of even degrees other than 4.

For regular graphs of odd degrees the only known reduction (construction) procedure is that for cubic graphs. While it can be easily seen that H-type reductions reduce some of them, whether they are sufficient is anyone's guess.

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The proofs of Lemma 1 and Case 2 of Theorem 1 are by one of the reviewers and are considerably more elegant and simpler than the original ones by the author. Also pointed out to the author by the same reviewer are Remarks 1 and 2, which greatly simplify the proof of Theorem 3.

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